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A. S. Cattaneo and G. FelderOn the Globalization of Kontsevich's Star Product and the Perturbative Poisson Sigma Model

On the Globalization of Kontsevich's Star Product and the Perturbative Poisson Sigma Model

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The globalization of Kontsevich's local formula (resp., the perturbative expansion of the Poisson sigma model) is described in down-to-earth terms.

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Introduction The problem of deformation quantization consists of deforming, in the realm of associative algebras, the pointwise product of smooth functions on a smooth manifold M in the direction of a given Poisson bracket (plus the conditions that the new product is defined in terms of bidifferential operators that kill constants). In the case when M is ^{n}, KontsevichK produced a remarkable formula in terms of the Poisson bivector field α that generates the Poisson bracket. This formula can also be viewedCF1 as the perturbative expansion of a certain expectation value in the so-called Poisson sigma modelI,SS with target $(^n, \alpha)$ and worldsheet a disk. As the formula transforms in a very complicated way under diffeormorphisms, it is a nontrivial task to get a global formula for a generic Poisson manifold (M, α). This has been described in K in terms of formal geometry and made explicit in CFT (see also CFT1/2).

The first aim of this paper is to present in down-to-earth terms the globalization of Kontsevich's local formula. Its second aim is to (start to) understand this globalization in terms of the Poisson sigma model.

Our exposition is based on Weinstein's approachW and on the results of CFT (rather than on Kontsevich'sK). Namely, we proceed as follows. We identify—e.g., by considering the exponential map ϕ for a torsion-free connection—a neighborhood U of the zero section of the tangent bundle TM at each point x with a neighborhood of x in M. This way we can use Kontsevich's formula fiberwise on the tangent bundle. More precisely, we can express the functions f and g to be multiplied and the Poisson bivector field α as objects living on T_xM and use Kontsevich's formula to get a new function on T_xM. As this has to be repeated for every point $x \in M$, the result will be actually a function $\sigma_{f,g}$ on U. Restricting this function to the zero section yields finally a new function $f \bullet g$ on M, which we may interpret as the product of f and g (see subsection ssec-assoc). This product is a deformation of the pointwise product along the direction of the Poisson bracket; it is however in general not associative. One may however observe that the product would be indeed associative if, for every pair of functions f and g on M, the corresponding function $\sigma_{f,g}$ on U were the pullback by ϕ of a function on M; for in this case the restriction to the zero section would be the inverse of the pullback by ϕ, and associativity of the global product would be an immediate consequence of the associativity of Kontsevich's product on each fiber. We may then try to modify Kontsevich's product to an equivalent one on each fiber (with possibly different equivalences on different fibers), so that the above lucky situation actually occurs. We call "quantization map" such a family of equivalences. Fortunately, it is possible to prove that quantization maps exist.CFT

Before delineating the proof, we must recall that Kontsevich's formula actually depends only on the Taylor expansions at zero of the functions to be multiplied and of the Poisson bivector field. In our case, we have to Taylor expand around the zero section of TM the pullbacks by ϕ of global functions. This way we obtain particular sections of the the jet bundle E, see subsection ss-Gc. (We may think of sections of E as of functions on an infinitely small neighborhood tM of the zero section of TM.)

The proof of the existence of quantization maps is in three steps. First, we recall (see subsection ss-Gc) that sections of E corresponding to global functions on M (as pullbacks by ϕ to tM) are in one-to-one correspondence with horizontal sections for a flat connection D on E. Next (see Sect. sec-dqjb), we use Kontsevich's formality theorem (reviewed in Sect. sec-kf) to deform D to a new connection that is a derivation for the fiberwise product and then use cohomological arguments to show that it is possible to further deform to a flat connection $^-$ that is still a derivation. Finally (see Sect. sec-dqPm), again by cohomological arguments, we prove that there is no obstruction in finding an isomorphism of (formal power series in the deformation parameter of) sections of E that intertwines between D and$^-$. This isomorphism is precisely the quantization map we were looking for.

We conclude the paper (see Sect. sec-cpePsm) by analyzing the above construction in terms of the Poisson sigma model. We observe that an exponential map ϕ may be used to define a change of coordinates in the functional integral that, up to problems on the boundary, preserves the functional measure and the BV bracket. In the new coordinates and ignoring the boundary problems, the perturbative expansion yields the globally defined non-associative $\bullet$-product described above (and in subsection ssec-assoc). It would be very interesting to understand how (and if) the correct treatment of the boundary produces a quantization map that yields an associative, global star product.

Kontsevich's formula and formality mapsec-kf We recall here the definitionK of the formality maps U (which can also be regardedCF1 as expectation values of the Poisson sigma model).

Given a collection $\xi_1, \ldots, \xi_n$ of multivector fields on ^{d} of degrees $k_1, \ldots, k_n$, one defines the multidifferential operator $U_n(\xi_1, \ldots, \xi_n)$, which acts on $\ell := 2 - 2n + \sum_{i=1}^n k_i$ functions, as follows: Let

$G_{k_1,\dots,k_n;\ell}$ denote the set of graphs with $\sum_{i=1}^n k_i + \ell$ numbered vertices such that the jth vertex for $j \leq \sum_{i=1}^n k_i$ emanates exactly k_j arrows (with the condition that no arrow ends where it begins). Then $\mathrm{U}_n(\xi_1,\dots,\xi_n) := \sum_{\Gamma\in G_{k_1,\dots,k_n;\ell}} w_\Gamma\, D_\Gamma, where \mathrm{D}_\Gamma$ is the multidifferential operator obtained by putting the multivector field ξ_j on the jth vertex and interpreting each arrow as a partial derivative. The weights w_Γ are obtained by certain integrals. We remark that ℓ is defined so that w_Γ vanishes for $\Gamma \in G_{k_1,\dots,k_n;r}$ with $r \neq \ell$. The formality theoremK states that the Us satisfy certain quadratic relations (which can be regardedCF1 as Ward identities for the Poisson sigma model).

We are interested only in particular cases of the above formulae; viz., when all but at most two of the multivector fields are equal to a given bivector field α and the remaining (zero, one or two) multivector fields are vector fields (which we denote momentarily by the letters ξ and ζ). Then we define subeqnarrayPAF $\mathrm{P}(\alpha) = \sum_{j=0}^\infty \epsilon^j j!\, U_j(\alpha,\dots,\alpha), P$

From now on, we assume that α is Poisson. Then the formality theorem ensures that $f \star g := P(\alpha)(f \otimes g)$ defines an associative product on $C^\infty(^d)[[\epsilon]]$. This star product is a deformation quantization of the commutative pointwise product along the direction of the Poisson structure α, as follows from Pe. Consider now a vector field ξ and its flow Φ_t. Define $f \star_t g := P(\Phi_{t*}\alpha)(f \otimes g)$. Then the formality theorem implies equationAP $\mathrm{A}(\xi,\alpha) f \star g + f \star A(\xi,\alpha) g - A(\xi,\alpha)(f \star g) = t\big|_{t=0}(f \star_t g)$.